Quantum Graphs & Quantum Random Walks
in Mathematics, Computer Science and Physics

Lake Como School of Advanced Studies, 5-9 August 2019

Courses

I. **Andris Ambainis**
Quantum Random Walks in Physics and Computer Science

II. **Gregory Berkolaiko**
Spectral Theory of Quantum Graphs

III. **Diego Noja**
Nonlinear Waves on Quantum Graphs

IV. **Uzy Smilansky**
Graphs and their Applications in Quantum Chaos

V. **Simone Warzel**
Anderson Localization in Graphs

Seminar Talks

- Lior Alon
A Central Limit Type Conjecture for the Nodal Statistics of Quantum Graphs
- Sebastian Egger
Bound States of a Pair of Particles on the Half-line with General Interaction Potential
- Latif Eliaz
On Graph Limits and the Essential Spectrum of Schrödinger Operators
- Charlie Johnson
Generalised Time-reversal Symmetries on Quantum Graphs
- Guillaume Lévy
On Quantum Graphs which Optimize their Spectral Gap
- Gökhan Mutlu
On the Quotient Graph with Respect to the Regular Representation
- P.A. Narayanan
Eigenvalue Statistics for Higher Rank Anderson Tight Binding Model over the Canopy Tree
- Selim Sukhtaev
Localization for Anderson Models on Metric and Discrete Tree Graphs

Tabular programme

	Mon 5.8.2019	**Tue 6.8.2019**	**Wed 7.8.2019**	**Thu 8.8.2019**	**Fri 9.8.2019**
8:45	Welcome				
9:00	**II.** Lecture 1	**II.** Lecture 3	**III.** Lecture 4	**I.** Lecture 3	**II.** Lecture 5
10:00	**II.** Lecture 2	**IV.** Lecture 2	**IV.** Lecture 3	**I.** Lecture 4	Seminar Talks
11:00	Coffee	Coffee	Coffee	Coffee	Coffee
11:30	**IV.** Lecture 1	**V.** Lecture 3	**I.** Lecture 1	**V.** Lecture 5	**IV.** Lecture 4
12:30	Seminar Talks	**V.** Lecture 4	**I.** Lecture 2	**III.** Lecture 5	**IV.** Lecture 5
13:30	Lunch	Lunch	Lunch	Lunch	Lunch
14:30	**V.** Lecture 1	**III.** Lecture 2	Free	**II.** Lecture 4	
15:30	**V.** Lecture 2	**III.** Lecture 3	Free	Seminar Talks	
16:30	Coffee	Coffee	Free	Coffee	
17:00	**III.** Lecture 1	Poster Session	Free	Seminar Talks	
19:00		Dinner			

Location

All lectures and seminar talks will be held at the Villa del Grumello, Via per Cernobbio 11, Como.

Detailed Programme

Sunday, 4 August

From 19:00
“Aperitivo” at the Bistrot Muralto (details send to particpants by email).

Monday, 5 August

From 8:00
Registration at Villa del Grumello

8:45
Welcome address

9:00
Gregory Berkolaiko, *Spectral Theory of Quantum Graphs*, lectures 1 & 2

11:00
Coffee break

11:30
Uzy Smilansky, *Graphs and their Applications in Quantum Chaos*, lecture 1

12:30
Guillaume Lévy, *On Quantum Graphs which Optimize their Spectral Gap*, seminar talk (25 minutes + 5 minutes discussion)

13:00
Sebastian Egger, *Bound States of a Pair of Particles on the Half-line with General Interaction Potential*, seminar talk (25 minutes + 5 minutes discussion)

13:30
Lunch

14:30
Simone Warzel, *Anderson Localization in Graphs*, lectures 1 & 2

16:30
Coffee break

17:00
Diego Noja, *Nonlinear Waves on Quantum Graphs*, lecture 1

Tuesday

9:00
Gregory Berkolaiko, *Spectral Theory of Quantum Graphs*, lecture 3

10:00
Uzy Smilansky, *Graphs and their Applications in Quantum Chaos*, lecture 2

11:00
Coffee break

11:30
Simone Warzel, *Anderson Localization in Graphs*, lectures 3 & 4

13:30
Lunch

14:30
Diego Noja, *Nonlinear Waves on Quantum Graphs*, lectures 2 & 3

16:30
Coffee break

From 17:00
Poster session with a wine reception at about 17:30.

19:00
Buffet Dinner at the Villa del Grumello

Wednesday

9:00
Diego Noja, *Nonlinear Waves on Quantum Graphs*, lecture 4

10:00
Uzy Smilansky, *Graphs and their Applications in Quantum Chaos*, lecture 3

11:00
Coffee break

11:30
Andris Ambainis, *Quantum Random Walks in Physics and Computer Science*, lecture 1 & 2

13:30
Lunch

From 14:30
Free afternoon to explore Como, Lake Como and the surrounding mountains. Suggestions for trips will be announced during the school.

Thursday

9:00
Andris Ambainis, *Quantum Random Walks in Physics and Computer Science*, lecture 3 & 4

11:00
Coffee break

11:30
Simone Warzel, *Anderson Localization in Graphs*, lecture 5

12:30
Diego Noja, *Nonlinear Waves on Quantum Graphs*, lecture 5

13:30
Lunch

14:30
Gregory Berkolaiko, *Spectral Theory of Quantum Graphs*, lecture 3 & 4

15:30
P.A. Narayanan, *Eigenvalue Statistics for Higher Rank Anderson Tight Binding Model over the Canopy Tree*, seminar talk (25 minutes + 5 minutes discussion)

16:00
Selim Sukhtaev, *Localization for Anderson Models on Metric and Discrete Tree Graphs*, seminar talk (25 minutes + 5 minutes discussion)

16:30
Coffee break

17:00
Gökhan Mutlu, *On the Quatient Graph with Respect to the Regular Representation*, seminar talk (25 minutes + 5 minutes discussion)

17:30
Charlie Johnson, *Generalised Time-reversal Symmetries on Quantum Graphs*, seminar talk (25 minutes + 5 minutes discussion)

Friday

9:00
Gregory Berkolaiko, *Spectral Theory of Quantum Graphs*, lecture 5

10:00
Lior Alon, *A Central Limit Type Conjecture for the Nodal Statistics of Quantum Graphs*, seminar talk (25 minutes + 5 minutes discussion)

10:30
Latif Eliaz, *On Graph Limits and the Essential Spectrum of Schrödinger Operators*, seminar talk (25 minutes + 5 minutes discussion)

11:00
Coffee break

11:30
Uzy Smilansky, *Graphs and their Applications in Quantum Chaos*, lectures 4 & 5

13:30
Lunch & end of school

Seminar Talk Abstracts

Lior Alon (Technion, Haifa)

A central limit type conjecture for the nodal statistics of quantum graphs

The nodal distribution of a given standard quantum graph have been shown to hold information about the topology of the graph, and it was explicitly calculated for specific families of graphs. In all of those cases, and in any numerical simulation, the nodal statistics appears to obey a central limit type convergence to a normal distribution as the number of edges (more specifically, the first Betti number) goes to infinity. We conjecture that this central limit type convergence of the nodal statistics is a universal property of quantum graphs. In the talk I will define the nodal statistics, state the conjecture and describe the proof of convergence for specific families of graphs. This talk is based on Joint works with Ram Band and Gregory Berkolaiko.

Sebastian Egger (Technion, Haifa)

Bound states of a pair of particles on the half-line with a general interaction potential

We study an interacting two-particle system on the half-line. We focus on spectral properties of the Hamiltonian for a large class of two-particle potentials. We characterize the essential spectrum and prove, as the main result, the existence of eigenvalues below the bottom of it. We also prove that the discrete spectrum contains only finitely many eigenvalues.

Latif Eliaz (Technion, Haifa)

On graph limits and the essential spectrum of Schrödinger operators

It is known that the essential spectrum of a Schrödinger operator H on $\ell^2(\mathbb{N})$ is equal to the union of the spectra of right limits of H. The natural generalization of this relation to $\mathbb{Z}^n$ is known to hold as well. In this talk we study the possibility of generalizing this characterization of $\sigma_{ess}(H)$ to graphs. The natural generalization of the concept of right limits to graphs turns out to be a Schrödinger operator version of Benjamini-Schramm limit. We show that the general characterization of the essential spectrum fails, while presenting natural families of models where it still holds.

Charlie Johnson (University of Bristol)

Generalised Time-Reversal Symmetries on Quantum Graphs

A major result of quantum chaos is that energy level sub spectra of chaotic systems correspond to random matrix ensembles. Furthermore, these ensembles can be completely determined by studying the symmetries of the system, of both unitary (geometric) and anti- unitary (time-reversal) type. It is then useful to be able to construct systems with arbitrary unitary and anti-unitary symmetries to test the random matrix theory ensembles. Quantum Graphs have typically been used for this purpose, but they have so far been restricted to containing only a simple complex-conjugation time-reversal symmetry, or completely broken time-reversal symmetry. We demonstrate how to generate graphs with generalised time-reversal symmetries, combining complex-conjugation with a unitary component, and show these graphs fulfil their random matrix theory predictions.

Guillaime Lévy (École polytechnique, Palaiseau Cedex)

On quantum graphs which optimize their spectral gap

In this talk, we consider the 1D Laplacian acting on functions defined on a metric graph. The vertex conditions, which are a replacement for the classical notion of a boundary condition in PDEs, are chosen to be Neumann everywhere hence enforcing the conservation of total current through a vertex. Given a graph, we are allowed to play with the lengths of its edges while keeping their sum fixed. Under these constraints, we strive to find the extremal values of the forst nonzero eigenvalue, equal to the spectral gap for the Neumann conditions and to relate it to other intrinsic properties of the graph (number of edges, of loops, of leaves, of cycles for instance). Moreover, we would like to know the possible shapes of extremizers. In collaboration with Ram Band (Technion), we have solved the minimization problem entirely, giving sharp bounds and characterizing the possible graphs which may arise as minimisers. Regarding the maximum problem, we only have partial answers; we also provide some tools and ideas to tackle the general problem. We conclude by stating a conjecture on the possible shapes of the maximizers.

Gökhan Mutlu (Gazi university, Ankara)
On the quotient quantum graph with respect to the regular representation
Joined work with Ercan Altınışık
Given a quantum graph Γ, a finite symmetry group G acting on it and a repre- sentation R of G, the quotient quantum graph Γ/R is described and constructed in literature. Band et al. constructed a pair of isospectral graphs in [3] and then in [2, 4] the authors generalized this construction and provided a rigorous recipe to construct quotient quantum graphs by using linear representations. Namely, given a quantum graph Γ, a symmetry group G acting on Γ and a representation R of G, they contructed another quantum graph Γ/R called a quotient quantum graph. This construction is based on an " encoding scheme" in which one builds the eigenfunctions f on Γ/R with the information on the eigenfunctions $\tilde{f}$ living on the original graph Γ such that $\tilde{f}$ transform according to R, i.e. $g\tilde{f} = R(g)\tilde{f}$ for all $g \in G$. Recently, Band et al. revised this construction method for quantum graphs and even applied to abstract finite dimensional operators to obtain quotient operators of such operators [1]. They also obtained quotients of discrete graphs. In [4], it was shown that for a quantum graph Γ and a symmetry group G acting on Γ, the quotient graph Γ/CG is isospectral to Γ where CG denotes the regular representation of G. The authors presented an open question: "is Γ/CG identical to Γ?"[4]. We answer the above open question affirmatively. For a quantum graph Γ and a finite symmetry group G acting on it, we construct the quotient quantum graph Γ/CG and show that the resulting graph is exactly the original graph Γ. We take a further step and show that, if one construct the quotient graph Γ/ρ, where ρ is an arbitrary permutation representation of G with degree $|G|$, one gets exactly the original graph Γ.
[1] Band R, Berkolaiko G, Joyner C H and Liu W 2018 Quotients of finite-dimensional operators by symmetry representations arXiv:1711.00918v3
[2] Band R, Parzanchevski O and Ben-Shach G 2009 The isospectral fruits of representation theory: quantum graphs and drums J. Phys. A: Math. Theor. 42 175202
[3] Band R, Shapira T and Smilansky U 2006 Nodal domains on isospectral quantum graphs: the resolution of isospectrality? J. Phys. A: Math. Gen. 39 13999
[4] Parzanchevski O and Band R 2010 Linear representations and isospectrality with boundary conditions, J. Geom. Anal. 20 439-71

P.A. Naryanan (Insititute of Mathematical Sciences, Chennai)
Eigenvalue statistics for higher rank Anderson tight binding model over the canopy tree
Here, we study the local eigenvalue statistics of the Anderson tight binding model with non-rank-one perturbations over the canopy tree, at large disorder. Canopy tree is an infinite single ended tree, with homogeneous degree except for its boundary vertices. Intuitively, canopy tree is like an inverted Bethe lattice. On the Hilbert space $\ell^2(\mathcal{C})$, where $\mathcal{C}$ is the canopy graph, we consider the random operator $\Delta_{\mathcal{C}} + \sum_{y\in J} \omega_y P_y$, where $\Delta_{\mathcal{C}}$ is the adjacency operator of the canopy tree, $\{\omega_y\}_{y\in J}$ are independentally and identically distributed real random variables following an absolutely continuous distribution μ which has a bounded density ρ with compact support, J is a suitable indexing set, P_y 's are projections on $\ell^2(\{x \in \mathcal{C} : d(y, x) < m_0 \text{and} y \prec x\})$. Here, $m_0 \geq 2$ is an arbitrarily fixed integer, by $y \prec x$ we mean both y and x are in the same unique path from the boundary of the canopy tree and that y is situated closer to the boundary, in this path, than x. For this operator we show that the eigenvalue-counting point process converges to a non-trivial compound Poisson process. To show the non-trivial nature of the compound Poisson process we use an observation in the joint work [2] with Anish Mallick that the higher rank canopy Hamiltonian has, almost surely, eigenvalues of higher multiplicity (which depends on the size of the perturbing projections, P_y 's). Aizenman and Warzel in the paper [1] showed that for the Anderson tight binding model with rank-one perturbations over the canopy tree, the eigenvalue-counting statistics is Poissonian. The Poisson nature of the eigenvalue statistics is usually thought to be associated to the localized regime of the spectrum. But the Anderson tight binding model on the Bethe lattice shows non-localized regime (ac-spectrum), but its eigenvalue counting statistics is, always, Poissonian, as shown in [1]. Aizenman and Warzel [1] explained the phenomena by showing that it is the canopy Hamiltonian which captures the limit of the eigenvalue-counting point process and that the canopy Hamiltonian shows spectral localization through out its spectrum, for any strength of disorder.
[1] Michael Aizenman and Simone Warzel. The canopy graph and level statistics for random operators on trees. Mathematical Physics, Analysis and Geometry, 9(4):291–333, 2006.
[2] Anish Mallick, Narayanan P. A. On multiplicity of spectrum for Anderson type opera- tors with higher rank perturbations. Arxiv Preprint arXiv:1808.05820 , August, 2018.

Selim Sukhtaev (Rice University, Houston)

Localization for Anderson models on metric and discrete tree graphs

This talk concerns Anderson localization for several random models on metric and combinatorial graphs. I will show that Anderson localization for random branching, random lengths, and random Kirch- hoff models holds outside of a discrete set of exceptional energies.